y = distance coordinate normal to surface L

Greek Letters

α = angle of separation of forward flow

β = coefficient of cubical expansion θ^{-1}

δ = boundary layer thickness L

η = dimensionless concentration gradient defined in text

ϵ = angle between normal to surface and the vertical

θ = angle from forward stagnation point

ρ_s = solid density ML^{-3}

ρ = fluid density ML^{-3}

$\Delta\rho$ = density difference across boundary layer ML^{-3}

σ = standard deviation

ν = kinematic viscosity L^2T^{-1}

Dimensions

L = length

M = mass

T = time

θ = temperature

Dimensionless Groups

N_{Gr} = $(gd^3\beta\Delta T)/(\nu^2)$ = Grashof number for heat transfer

N'_{Gr} = $(gd^3\Delta\rho)/(\nu^2\rho)$ = Grashof number for mass transfer

N_{Pr} = ν/k = Prandtl number

N'_{Ra} = $(gd^3\Delta\rho)/(\nu^2\rho)\ \nu/D$ = Rayleigh number

N_{Re} = Reynolds number based on average duct velocity

N'_{Re} = Reynolds number based on approach velocity

N_{Re}^{+} = equivalent Reynolds number defined in text

$N_{Re_{w}\,max}$ = free convective Reynolds number defined in text

N_{Sc} = ν/D = Schmidt number

N_{Sh} = K_Ld/D = Sherwood number

Superscribed bars indicate over-all values.

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A Theoretical Analysis of Laminar Natural Convection Heat Transfer to Non-Newtonian Fluids

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An equation is given for the local Nusselt number in laminar convection heat transfer to power-law non-Newtonian fluids. This expression, which even for Newtonian fluids ($n = 1$) does not appear to have been derived before, is obtained from the exact asymptotic solution of the appropriate laminar boundary layer equations and is applicable to any two-dimensional surface or a surface of revolution about an axis of symmetry when, as is usually the case, $N_{Pr} > 10$.

The increasing emergence of non-Newtonian fluids, such as molten plastics, pulps, emulsions, etc. as important raw materials and products in a large variety of industrial processes, has stimulated a considerable amount of interest in the behavior of such fluids when in motion. In particular what has been studied most intensely for obvious practical reasons is how momentum and heat are transferred to a moving non-Newtonian fluid under the more common flow configurations usually met in practice. It is understandable therefore that some of the simpler problems of classical hydrodynamics, such as pressure drop and heat transfer in pipes and channels, flow between rotating concentric cylinders, etc., have been reinvestigated by the use of many types of non-Newtonian fluids. The results of these studies are to be found in the review article by Metzner (*9*) and in some recent publications on this subject (*3, 4, 10*).

In keeping with this general trend Acrivos, Petersen, and Shah (*1*) have recently presented a theoretical analysis of forced convection momentum and heat transfer in laminar boundary-layer flows of non-Newtonian fluids past external surfaces. As is well known, laminar boundary-layer theory, in which the viscous terms of the equations of motion are retained only in a very thin region near the surface, has made possible the study of a rather general and important class of fluid mechanical systems. It appeared worthwhile therefore to extend the boundary-layer theory to non-Newtonian fluids in order to investigate even further their properties under flow conditions.

The present analysis may be considered as a direct continuation of the work reported earlier (*1*). Its purpose is to study theoretically the problem of

It is interesting to note that Equations (27) and (36) can readily be integrated over the surface to yield the average Nusselt number $N_{\overline{Nu}}$. Thus from Equation (36)

$$N_{\overline{Nu}} \int_0^{x_1} r_1\, dx_1 = -\theta'(0)\left(\frac{3n+1}{2n+1}\right)^{\frac{2n+1}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}}\, N_{Pr}^{\frac{n}{3n+1}} \left[\int_0^{x_1} r_1^{\frac{3n+1}{2n+1}} (\sin\epsilon)^{\frac{1}{2n+1}}\, dx_1\right]^{\frac{2n+1}{3n+1}}$$

natural convection heat transfer to non-Newtonian fluids and to show how the well-established expressions for the rate of heat transfer to Newtonian fluids may be generalized to include the non-Newtonian effects. Again the laminar boundary-layer equations will be extended to and solved for the power-law non-Newtonian fluids, and the analysis will apply to the flow past arbitrary two-dimensional surfaces or to surfaces of revolution about an axis of symmetry.

LAMINAR BOUNDARY-LAYER EQUATIONS FOR POWER-LAW FLUIDS

The term "non-Newtonian fluid" is one of very great generality and includes all fluids for which the equations of classical hydrodynamics do not apply. As a consequence a chief difficulty in any theoretical analysis of the motion of such fluids has always been the lack of any generally acceptable equation of state between the stress tensor and the state of flow of the system. It is fortunate however that in a rather sizeable class of non-Newtonian fluids the stress-strain-velocity relations do not involve time derivatives of the stress- or the strain-velocity components and may therefore be represented under isotropic conditions by Reiner's (*16*) or Rivlin's (*17*) invariant expressions. It was explained in the earlier paper (*1*) furthermore that for one-dimensional or boundary-layer types of flow these fluids may often be characterized with satisfactory accuracy by the empirical power-law

$$\tau = K \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \tag{1}$$

where K and n are empirical constants characteristic of the fluid, and τ and $\partial u/\partial y$ are the only components of the stress tensor and the deformation tensor which need to be considered under such flow conditions. Thus, although strictly speaking Equation (1) should be replaced by the more general form of the power law which has been proposed by Mooney and Black (*11*) and others, it can be shown rigorously that for boundary-layer flows under either forced or free convection the term $\partial u/\partial y$ is so much larger than all the other elements in the deformation tensor that the one-dimensional power law shown above is in general quite adequate. The present analysis will therefore be restricted to those systems which satisfy Equation (1).

Consider now the flow past the arbitrary two-dimensional isothermal surface of temperature T_s shown in Figure 1. The laminar boundary-layer equations for the natural convection of Newtonian fluids may be found in the standard references (*6, 7, 18*), and their extension to non-Newtonian fluids can easily be derived from the analogous forced convection analysis already reported (*1*). Thus if for simplicity constant properties are postulated, except of course for the density in the buoyancy term, and if, as is usually permissible to a first approximation, the frictional dissipation term in the energy equation is neglected, one can show that

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g\beta(T_s - T_\infty)\theta \sin \epsilon + \frac{K}{\rho} \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y} \left| \frac{\partial u}{\partial y} \right|^{n-1} \right] \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 \theta}{dy^2} \tag{4}$$

where the symbols have the usual meaning (see Figure 1). β is defined by

$$\frac{\rho_\infty}{\rho} = 1 + \beta(T - T_\infty) \tag{5}$$

The boundary conditions are

$$\begin{aligned} &\text{At } y = 0,\ u = 0,\ v = 0,\ \theta = 1 \\ &\text{At } y = \infty \text{ and at } x = 0,\ u = 0 \\ &\text{and } \theta = 0^* \end{aligned} \tag{6}$$

One rather important characteristic of the power law must be brought out at this point. It should be firmly realized that, as has been repeatedly verified by experiments (*2, 8*), all fluids with time-independent stress properties approach Newtonian behavior when all the components of the deformation tensor become small. Thus for some fluids Equation (1) does not become valid until $\partial u/\partial y$ is larger than 100 sec.$^{-1}$, whereas for other systems the power law is found to hold for a rate of strain as low as 10^{-2} sec.$^{-1}$ or even lower (*12*). In particular, since in natural convection the velocity gradients usually encountered are of the order of magnitude of 0.1-1.0 sec^{-1}, one must keep in mind that for the purposes of the present analysis a fluid should be considered non-Newtonian only if $\partial u/\partial y$ at the surface has a numerical value which is within the region of applicability of the power law by at least an order of magnitude. This would then insure a non-Newtonian behavior on the part of the fluid within the main part of the boundary layer. Otherwise many systems which would normally obey the power law in forced convection might very well behave as Newtonians in natural convection flows.

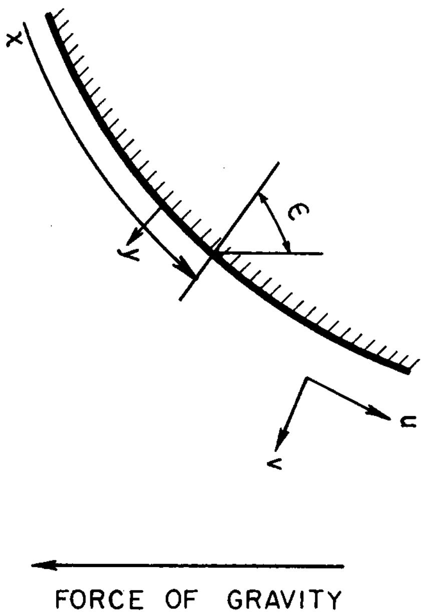

Fig. 1. The position directions of the coordinates x and y, the velocity components u and v, and the angle ϵ.

It is possible to show now, by a straightforward analysis of the basic Equations (2) to (4), that the familiar definitions for the Grashof and the Prandtl number, which play such a fundamental role in natural convection phenomena, may be generalized to power law non-Newtonian fluids so that

$$N_{Gr} = \frac{\rho^2 L^{n+2}[\beta g(T_s - T_\infty)]^{2-n}}{K^2} \tag{7}$$

and

$$N_{Pr} = \frac{\rho c_p}{K}\left(\frac{K}{\rho}\right)^{\frac{2}{1+n}} (L)^{\frac{1-n}{1+n}} [L\,\beta g(T_s - T_\infty)]^{\frac{3(n-1)}{2(1+n)}} \tag{8}$$

However these relations, which of course reduce to the usual expressions for Newtonian fluids when $n = 1$, may also be derived by the following rather interesting argument. It has already been shown (*1*) that in forced convection to power-law fluids the generalized Reynolds and Prandtl numbers are, respectively

$$N_{Re} \equiv \frac{\rho U_0^{2-n} L^n}{K} \tag{9}$$

* The proper boundary condition for θ at the leading edge ($x = 0$) has to be stated with some care, for it is intimately tied up with the normal velocity v at $x \to 0$. Thus if at the leading edge $v \to \infty$ for all y, then clearly $\theta = 0$, since the fluid in this region is continually renewed with fresh material from the bulk at a very high rate. Conversely if $v \to 0$ for all y and $x \to 0$, then the boundary condition is $\theta = 1$, since the region near the leading edge consists essentially of stagnant fluid. Incidentally the same difficulty arises when the proper boundary condition for forced convection heat transfer is specified as well.

and

$$N_{Pr} \equiv \frac{c_p \, \rho \, L \, U_c}{k \, N_{Re}^{\frac{2}{1+n}}} * \qquad (10)$$

For free convection the characteristic velocity is always related to the buoyancy forces by means of

$$U_c = \sqrt{L g \, \beta (T_s - T_\infty)} \qquad (11)$$

and if therefore this expression for U_c is substituted in the Reynolds and Prandtl numbers defined above, Equations (7) and (9) become

$$N_{Re} = \sqrt{N_{Gr}} \qquad (12)$$

while Equations (8) and (10) are seen to be identical. It follows then that since Equations (11) and (12) are perfectly general, as can be readily shown from the basic equation of motion, Equation (2), and since Equations (9) and (10) have already been established for forced convection, the definition of the Grashof and Prandtl numbers given respectively by Equations (7) and (8) follows as a logical consequence.

Having thus defined the characteristic velocity for the process, Equation (11), and the generalized groups N_{Gr} and N_{Pr}, one is now in a position to simplify the boundary-layer equations by a transformation in the position coordinates and the velocity components. Let the new variables be

$$x_1 = \frac{x}{L}, \quad u_1 = \frac{u}{\sqrt{Lg\beta(T_s - T_\infty)}},$$

$$y_1 = \frac{y}{L} N_{Gr}^{\frac{1}{2(n+1)}},$$

$$v_1 = \frac{v}{\sqrt{Lg\beta(T_s - T_\infty)}} N_{Gr}^{\frac{1}{2(n+1)}} \qquad (13)$$

The boundary-layer equations then become

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} = \theta \sin \epsilon + \frac{\partial}{\partial y_1} \left[\frac{\partial u_1}{\partial y_1} \left| \frac{\partial u_1}{\partial y_1} \right|^{n-1} \right] \qquad (14)$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0 \qquad (15)$$

$$u_1 \frac{\partial \theta}{\partial x_1} + v_1 \frac{\partial \theta}{\partial y_1} = \frac{1}{N_{Pr}} \frac{\partial^2 \theta}{\partial y_1^2} \qquad (16)$$

with the boundary conditions

At $y_1 = 0$, $u_1 = 0$, $v_1 = 0$, $\theta = 1$

At $y_1 = \infty$ and at $x_1 = 0$, $u_1 = 0$,

* The author's definition of N_{Pr} differs from that used by Metzner (*10*). The two expressions are related by

$$N_{Pr} \text{ (Metzner)} = N_{Pr} \, (N_{Re})^{\frac{1-n}{1+n}}.$$

$$\theta = 0 \qquad (17)$$

It will be shown in the next section how this system of equations may be solved for any surface geometry, that is when $\sin \epsilon$ is an arbitrary function of x_1.

THE SOLUTION OF THE TRANSFORMED BOUNDARY-LAYER EQUATIONS

One first attempts to solve Equations (14) to (17) by means of a similarity transformation which is in keeping with a well-established tradition in boundary-layer theory (*6, 18*). It is known for example (*7, 18*) that for Newtonian fluids ($n = 1$) such a similarity transformation on the free convection equations may be performed if the surface geometry is of the form $\sin \epsilon = a \, x_1^m$ where a and m are arbitrary constants such that $a > 0$ and $m \geqq 0$.*

If $n \neq 1$ however, Equations (14) to (17) cannot be reduced in this manner unless $\sin \epsilon = a x_1^{-1/3}$, which obviously is unacceptable. It would appear therefore that the mathematical solution of the free convection problem to non-Newtonian fluids would be an almost impossible task to carry through, especially since it is quite clear that the basic partial differential equations are indeed highly complex in structure and cannot be solved, even numerically, except with a great deal of difficulty.

It turns out fortunately that the equations can be simplified still further, since for essentially all non-Newtonian fluids of interest the generalized Prandtl number N_{Pr} is much larger than unity. This allows us to investigate the solution of Equations (14) to (17), under the asymptotic condition $N_{Pr} \rightarrow \infty$, by an extension of the method used by Morgan and Warner (*13*) to solve analogous heat transfer problems in laminar boundary-layer flows to Newtonian systems. The appropriate transformations in the velocity components and the position coordinates are as follows. Let

$$y_2 \equiv y_1 \, N_{Pr}^{\frac{n}{3n+1}} = \frac{y}{L} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}}$$

$$u_2 \equiv u_1 \, N_{Pr}^{\frac{n+1}{3n+1}} = \frac{u}{\sqrt{Lg\beta(T_s - T_\infty)}} N_{Pr}^{\frac{n+1}{3n+1}}$$

* In accordance with Merk and Prins [Reference 7, Equation (16)] a similarity solution is possible if $\sin \epsilon = a \, (x_1 + b)^m$, where b is another constant. This appears incorrect, however, since the solution proposed by Merk and Prins [Equations (17) and (18)] does not satisfy the boundary conditions at the leading edge ($x_1 = 0$) unless $b = 0$.

$$v_2 \equiv v_1 \, N_{Pr}^{\frac{2n+1}{3n+1}} = \frac{v}{\sqrt{Lg\beta(T_s - T_\infty)}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{2n+1}{3n+1}} \qquad (18)$$

which when substituted into the modified boundary-layer Equations (14), (15), and (16) reduces them to

$$\frac{\partial}{\partial y_2} \left[\frac{\partial}{\partial y_2} \left| \frac{\partial u_2}{\partial y_2} \right|^{n-1} \right] + \theta \sin \epsilon = \frac{1}{N_{Pr}^{\frac{2(n+1)}{3n+1}}} \left\{ u_2 \frac{\partial u_2}{\partial x_1} + v_2 \frac{\partial u_2}{\partial y_1} \right\} \rightarrow 0$$

as $N_{Pr} \rightarrow \infty$

$$(19)$$

$$\frac{\partial u_2}{\partial x_1} + \frac{\partial v_2}{\partial y_2} = 0 \qquad (20)$$

$$\frac{\partial^2 \theta}{\partial y_2^2} = u_2 \frac{\partial \theta}{\partial x_1} + v_2 \frac{\partial \theta}{\partial y_2} \qquad (21)$$

The boundary conditions are the same as those given by Equation (17) except for one of the conditions at $y_2 = \infty$, which now reads $\partial u_2 / \partial y_2 = 0$ rather than $u_2 = 0$. Therefore exactly as is the case with free convection to Newtonian fluids the inertia terms [the right-hand side of Equation (19)] become asymptotically unimportant as $N_{Pr} \rightarrow \infty$ inside the thermal boundary layer and, as pointed out by Stewartson and Jones (*10*) for the analogous Newtonian heat transfer problem from a vertical flat plate, there are two distinct regions of flow when $N_{Pr} \rightarrow \infty$:

(1) The region inside the thermal boundary layer, which is described by the above equations with the inertia term in the equation of motion set equal to zero.

(a) The region outside the thermal boundary layer, where both θ and therefore the buoyancy force, but not the inertia terms, vanish. However since in all convection problems one is concerned only with the rate of heat transfer at the surface, which may be obtained from the solution of the equations in the thermal boundary layer, the velocity distribution outside this region is usually of no interest, and, as a consequence, need not be analyzed any further.

It is possible then to obtain the velocity and temperature distributions inside the thermal boundary layer by solving Equations (19), (20), and (21). Surprisingly enough this can be accomplished readily by a similarity transformation which even for $n = 1$

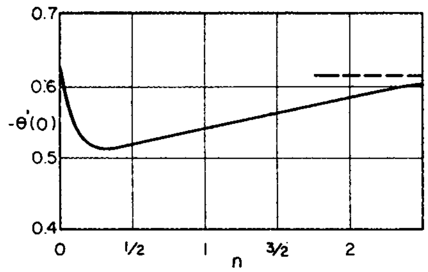

Fig. 2. The dependence of $-\theta'(0)$ on the non-Newtonian parameter n.

(Newtonian fluids) does not appear to have been given before. Thus if $\theta = \theta(\eta)$ and

$$u_2 = (\sin\epsilon)^{\frac{1}{n}} \left[\frac{3n+1}{2n+1}\left(\frac{1}{\sin\epsilon}\right)^{\frac{3n+1}{n(2n+1)}} \int_0^{x_1} (\sin\epsilon)^{\frac{1}{2n+1}} dx_1\right]^{\frac{n(n+1)}{3n+1}} f'(\eta) \tag{22}$$

where

$$\eta \equiv \frac{y_2}{\left[\frac{3n+1}{2n+1}\left(\frac{1}{\sin\epsilon}\right)^{\frac{3n+1}{n(2n+1)}} \int_0^{x_1} (\sin\epsilon)^{\frac{1}{2n+1}} dx_1\right]^{\frac{n}{3n+1}}} \tag{23}$$

one can show after making careful use of the fact that $f''(o) \gtreqq 0$ inside the thermal boundary layer that

$$\frac{d}{d\eta}(f'')^n + \theta = 0 \tag{24}$$

and

$$\theta'' + f\theta' = 0 \tag{25}$$

with the boundary conditions

$$\theta(0) = 1,\ \theta(\infty) = 0,\ f(0) = 0,$$
$$f'(0) = 0,\ f'(\infty) = 0$$

The two parameters of particular interest, $f''(0)$ and $\theta'(0)$, can finally be calculated without difficulty by the following simple scheme. Thus it is apparent from Equations (24) and (25) that formally

$$-\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = \frac{1}{\int_0^\infty e^{-\int_0^\eta f\,d\eta}\,d\eta} \text{ and}$$

$$\left[\frac{d^2f}{d\eta^2}\right]^n_{\eta=0} = \frac{\int_0^\infty \eta e^{-\int_0^\eta f\,d\eta}\,d\eta}{\int_0^\infty e^{-\int_0^\eta f\,d\eta}\,d\eta} \tag{26}$$

which can be computed quite readily bv substituting the power series of f about $\eta = 0$ and then evaluating the resulting integrals numerically. It is indeed rather surprising that, as can be seen from Figures 2 and 3, $\theta'(0)$ and $[f''(0)]^n$ are quite insensitive to the value of n in the range $0 < n \leqq 3/2$ which includes most non-Newtonian fluids of interest. In particular, for the special case $n = 1$, $\theta'(0) = 0.5404$ and $f''(0) = 1.08$.

In view then of Equations (13), (18), (23), and (27) the local Nusselt, which for heat transfer is defined as is customary by

$$N_{Nu} \equiv -L\left(\frac{\partial\theta}{\partial y}\right)_{y=0}$$

is given by

$$N_{Nu} = -\theta'(0)\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}} \frac{(\sin\epsilon)^{\frac{1}{2n+1}}}{\left[\int_0^{x_1} (\sin\epsilon)^{\frac{1}{2n+1}} dx_1\right]^{\frac{n}{3n+1}}} \tag{27}$$

asymptotically as $N_{Gr} \to \infty$ and $N_{Pr} \to \infty$, where the value of $\theta'(0)$, approximately equal to -0.54, may be read off Figure 2 as a function of n. It follows that the average Nusselt number $\overline{N_{Nu}}$ is given by

$$\overline{N_{Nu}} = C\, N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}} \tag{27a}$$

where the constant C can readily be calculated by integrating Equation (27) over the surface and would in general be expected to be a function not only of the surface geometry but also of n. It will be recognized right away that this final expression, Equation (27*a*), is the generalization to power law non-Newtonian fluids of the rather well-known correlation $\overline{N_{Nu}} = C(N_{Gr}N_{Pr})^{1/4}$ for Newtonian substances which has been repeatedly verified by experiments (*5, 7, 20*) and which, as demonstrated rigorously by Morgan and Warner (*13*), may be deduced from the laminar boundary-layer equations provided that $N_{Pr} >> 1$.

Equation (27) provides one with a closed-form expression for the local rate of heat transfer, in natural convection, from arbitrary two-dimensional surfaces to non-Newtonian fluids of the power law type. This solution is of course limited to those physical problems for which the simplifications introduced in the theory are realistic. This section is therefore concluded by a critical discussion of the main assumptions, listed below, which were used in the development.

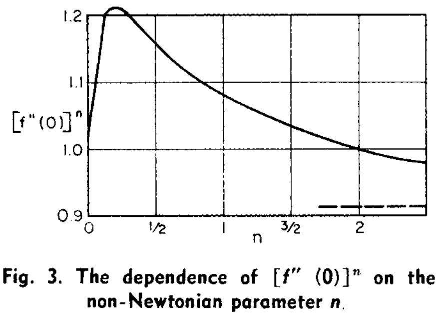

Fig. 3. The dependence of $[f''(0)]^n$ on the non-Newtonian parameter n.

(1) The laminar boundary-layer equation must be applicable. It is well known from boundary-layer theory that the boundary-layer equations are applicable only when the transfer of momentum and energy occurs in a very thin region near the surface. Let δ_T be the thickness of the thermal boundary layer which in view of Equations (13), (18), (22), and (23) is

$$\frac{\delta_T}{L} = 0\left\{\frac{1}{N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}}}\right\}$$

It follows, from a straightforward analysis of the basic equations, that the boundary-layer equations are valid if $\delta_T^2/L^2 << 1$, which incidentally is also the condition that allows the substitution of Equation (1) for the more general formulation of the power law given by Mooney and Black (*11*). From the expression for δ_T however this is equivalent to

$$N_{Gr}^{\frac{1}{n+1}} N_{Pr}^{\frac{2n}{3n+1}} >> 1^* \tag{28}$$

which, as can be seen from the following typical numerical example, is usually the case.

Consider the transfer in natural convection to a 0.83% solution of ammonium alginate in water, the properties of which are (*8*):

$n = 0.78$ $c_p = 1.0$ B.t.u./lbm. °F.
$K = 0.06$ lbm. sec.$^{n-2}$/ft.
$k = 0.37$ B.t.u./ft. °F. hr.
$\rho = 60$ lbm./cu. ft.

Let $T_s - T_w = 10$°F., $L = 1$ ft., and $\beta = 1.3 \times 10^{-4}$ (°F.)$^{-1}$. In view of Equations (7), (8), and (11)

$$U_o = 0.21 \text{ ft./sec.}$$
$$N_{Gr} \cong 2 \times 10^4$$
$$N_{Pr} \cong 440$$

and

$$N_{Gr}^{\frac{1}{(n+1)}} N_{Pr}^{\frac{2n}{3n+1}} \cong 4{,}600$$

which illustrates the point. One can also show that, for the present system at least, the use of the non-Newtonian power law model throughout the boundary layer is indeed justified. Thus, since $(\partial u_2/\partial y_2)_{y_2=0} \sim 1$, one can deduce from Equations (18) that the shear rate at the surface has a numerical value given by

* For a curved surface Equation (28) must be replaced by the more restrictive inequality

$$N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}} >> 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} \sim U_o \frac{N_{Gr}^{\frac{1}{2(n+1)}}}{N_{Pr}^{\frac{1}{3n+1}}} = 0.5 \text{ sec.}^{-1}$$

which appears to be within the range of applicability of Equation (1) with $n = 0.78$ (*8*).

2. The modified Prandtl number N_{Pr} must be large; otherwise the inertia terms in the equations of motion, Equation (19), cannot be neglected. It can be shown for example, by comparing the asymptotic solution with exact numerical computations carried out by Ostrach (*14*), that for the flow of a Newtonian fluid past a vertical flat plate Equation (27) is fairly accurate only if $N_{Pr} \geqq 10$. (Thus the error is 8.1 and 2.6% respectively when $N_{Pr} = 10$ and $N_{Pr} = 100$.) Fortunately however the generalized N_{Pr} for non-Newtonian fluids is usually quite large.

3. The fluid properties must be constant, a simplification which may introduce a nonnegligible error in the final result since most fluid properties are temperature dependent. This is especially true of the modified viscosity coefficient K. If however one is restricted to cases where $N_{Gr} >> 1$ and $N_{Pr} >> 1$, it can be shown that the constant property solution needs to be modified only slightly in order to be also applicable to this more general problem.

Let all the physical properties, ρ, K, c_p, k, β be referred to their respective values in the bulk, and let the dimensionless groups N_{Gr} and N_{Pr} be defined in terms of these constant bulk properties. Therefore if

$$\rho' \equiv \frac{\rho}{\rho_\infty},\ K' \equiv \frac{K}{K_\infty},\ c_p' \equiv \frac{c_p}{c_{p\infty}},$$
$$k' \equiv \frac{k}{k_\infty},\ \beta \equiv \frac{\beta}{\beta_\infty}$$

all of which are known functions of the temperature θ, it is possible to show that Equations (24) and (25) must be modified to

$$\frac{1}{\rho'}\frac{d}{d\eta}\left[K'\left(\frac{d^2f}{d\eta^2}\right)^n\right] + \beta'\theta = 0 \tag{29}$$

and

$$\frac{d}{d\eta}\left(k'\frac{d\theta}{d\eta}\right) + c_p'\frac{d\theta}{d\eta}\left[\int_0^\eta \rho'\frac{df}{d\eta}d\eta\right] = 0 \tag{30}$$

As was explained earlier the parameters of interest $\theta'(0)$ and $f''(0)$ can be calculated numerically without too much difficulty. Again Equation (27) will apply with the added complication however that now the parameter $\theta'(0)$ will also be influenced by the functional dependence of ρ, K, c_p, k, β on the temperature θ. Interestingly enough, therefore, the general expression for the Nusselt number given by Equation (27) will remain unchanged even when the properties are temperature dependent, provided that it is multiplied by a correction factor, which is independent of N_{Gr}, N_{Pr} and the surface geometry and is a function only of the particular fluid being used.

4. The final assumption of some importance is that the frictional heat-generation term in the energy equation is negligible. This is indeed a permissible simplification in many problems, since the velocities usually encountered in natural convection are rather small, but of course there are cases where such an assumption would cause an appreciable error in the calculated rate of heat transfer. Unfortunately the present analysis cannot be readily modified to this more general problem which must then be investigated by more elaborate techniques.

NATURAL CONVECTION PAST A FLAT PLATE AND A HORIZONTAL CYLINDER

Flow past a vertical flat plate

For a vertical flat plate $\sin \epsilon \equiv 1$, and therefore from Equation (27)

$$N_{Nu} = -\theta'(0)\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}} x_1^{\frac{-n}{3n+1}}$$

It follows that the average Nusselt number is

$$N_{Nu} = -\theta'(0)\left(\frac{3n+1}{2n+1}\right)^{\frac{2n+1}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}}$$

if the characteristic length L is set equal to the length of the plate.

Flow past an infinite, horizontal cylinder

For such a surface L = the radius of the cylinder and $\sin \epsilon = \sin x_1$, so that

$$N_{Nu} = -\theta'(0)\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}} \frac{(\sin x_1)^{\frac{1}{2n+1}}}{\left[\int_0^{x_1}(\sin x)^{\frac{1}{2n+1}}dx\right]^{\frac{n}{3n+1}}}$$

In the stagnation region $(0 \leqq x_1 \leqq \pi/6)$ $\sin x_1 \cong x_1$, and therefore

$$N_{Nu} \approx -\theta'(0)\left(\frac{2n+2}{3n+1}\right)^{\frac{n}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}} x_1^{\frac{1-n}{3n+1}}$$

For $x_1 > \pi/6$ it is necessary to evaluate the integral $\int_0^{x_1}(\sin x)^{\frac{1}{2n+1}}dx$ which may be done conveniently by transferring it into a tabulated (*15*) incomplete Beta function. Thus

For $0 \leqq x_1 \leqq \pi/2$

$$\int_0^{x_1}(\sin x)^{\frac{1}{2n+1}}dx = \frac{1}{2}\int_z^1 z^{-1/2}(1-z)^{\frac{-n}{2n+1}}dz$$

where $z \equiv \cos^2 x_1$.

For $\pi/2 < x_1 \leqq \pi$

$$\int_0^{x_1}(\sin x)^{\frac{1}{2n+1}}dx = \sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2n+1}\right)}{\Gamma\left(\frac{n+1}{2n+1}+\frac{1}{2}\right)} - \int_0^{\pi-x_1}(\sin x)^{\frac{1}{2n+1}}dx$$

A plot of $N_{Nu}/N_{Gr}^{\frac{1}{2(n+1)}}N_{Pr}^{\frac{n}{3n+1}}$ for the transfer of heat from a cylinder is shown in Figure 4 for $0 \leqq x_1 \leqq \pi$ and $n = 1/2$, 1, and 3/2.

FLOW PAST SURFACES OF REVOLUTION ABOUT AN AXIS OF SYMMETRY

It was first discovered by Mangler (*18*) that the boundary-layer equations for the forced-convection flow of a Newtonian fluid past a surface of revolution could be reduced by a coordinate transformation to a set of equations identical to those for two-dimensional flow. In this section the appropriate transformation for free convection heat transfer to power-law for non-Newtonian fluids shall be presented.

For the flow past three-dimensional surfaces of revolution Equations (2) and (4) and therefore Equations (19) and (21) still apply (*5, 11*). The continuity equation must however be modified to

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0 \tag{31}$$

where $r(x)$ is the distance of a point on the surface from the axis of symmetry. A simple transformation which enables one to reduce this three-dimensional problem to that already solved is given by

$$r_1 \equiv \frac{r(x_1)}{L};\ \bar{u}_2 \equiv \frac{u_2}{r_1};\ X = \int_0^{x_1} r_1(x_1)dx_1 \tag{32}$$

TABLE 1. NUMERICAL VALUES OF C FOR VARIOUS SURFACE GEOMETRIES AND n

	$n = 1/10$	$n = 1/2$	$n = 1$	$n = 3/2$
Flat plate	0.60	0.63	0.67[a]	0.71
Horizontal cylinder (L = radius)	0.36	0.38	0.42	0.45
Sphere (L = radius)	0.44	0.45	0.49[b]	0.52
Vertical cone	0.61	0.65	0.71	0.75
Stagnation region of a horizontal cylinder	0.36	0.45	0.54	0.60
Stagnation region of a sphere	0.45	0.55	0.64	0.70

[a] In exact agreement with the experimental value 0.66 (*20*).
[b] In exact agreement with the experimental value 0.49 (*5*).

It follows that the extension of Equations (19), (20), and (21) to the three-dimensional axisymmetric flow is, respectively

$$\frac{\partial}{\partial y_2}\left(\frac{\partial \overline{u}_2}{\partial y_2}\right)^n + \theta r_1^n \sin \epsilon(X) = 0 \tag{33}$$

$$\frac{\partial \overline{u}_2}{\partial X} + \frac{\partial v_2}{\partial y_2} = 0 \tag{34}$$

$$\frac{\partial_2 \theta}{\partial y_2^2} = \overline{u}_2 \frac{\partial \theta}{\partial X} + v_2 \frac{\partial \theta}{\partial y_2} \tag{35}$$

and therefore, in view of Equation (27) and the definition of X

$$N_{Nu} = -\theta'(0)\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}} \frac{(r_1^n \sin \epsilon)^{\frac{1}{2n+1}}}{\left[\int_0^{x_1} r_1^{\frac{3n+1}{2n+1}} (\sin \epsilon)^{\frac{1}{2n+1}} dx_1\right]^{\frac{n}{3n+1}}} \tag{36}$$

where again $\theta'(0)$, approximately equal to -0.54, may be read off Figure 2 as a function of n. Equation (36) is then the generalization of Equation (27) for three-dimensional axisymmetric flows and is applicable to any surface of revolution. Of course the average Nusselt number is again given by Equation (27*a*), where C must now be computed by properly averaging the local Nusselt number over the surface.

As a special case one should consider natural convection from a sphere. If the characteristic length is taken equal to the radius of the sphere, then

$$\sin \epsilon = \sin x_1 \text{ and } r_1 = \sin x_1$$

Therefore

$$N_{Nu} = -\theta'(0)\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}} \frac{(\sin x_1)^{\frac{n+1}{2n+1}}}{\left[\int_0^{x_1} (\sin x_1)^{\frac{3n+2}{2n+1}} dx_1\right]^{\frac{n}{3n+1}}} \tag{37}$$

where, as was explained in the previous section, the integral in the denominator can again be expressed in terms of the tabulated incomplete Beta function. For the front part of the sphere ($x_1 \to 0$) Equation (37) can readily be simplified to

$$N_{Nu} \to -\theta'(0)\left(\frac{5n+3}{3n+1}\right)^{\frac{n}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}} x_1^{\frac{1-n}{3n+1}} \text{ as } x_1 \to 0$$

which holds quite accurately in the range $0 \leq x_1 \leq \pi/6$. A plot of $N_{Nu}/N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3(n+1)}}$ for the transfer of heat from a sphere is shown in Figure 5 for $0 \leq x_1 \leq \pi$ and $n = 1/2$, 1, and 3/2.

On the other hand for natural convection from a vertical cone $\sin \epsilon$ = constant and $r_1 = x_1$. It is easy to see then that

$$N_{Nu} = -\theta'(0)\left(\frac{5n+2}{3n+1}\right)^{\frac{n}{3n+1}} x^{-\frac{n}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}}$$

and

$$\overline{N_{Nu}} = -2\theta'(0)\left(\frac{3n+1}{5n+2}\right)^{\frac{2n+1}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}}$$

if the characteristic length L is set equal to the length of the cone.

CONCLUDING REMARKS

The theoretical considerations of this paper show that in laminar natural convection heat transfer to power-law non-Newtonian fluids the local Nusselt number is given by

$$N_{Nu} = -\theta'(0)\left(\frac{2n+1}{3n+1}\right)^{\frac{n}{3n+1}} N_{Gr}^{\frac{1}{2(n+1)}}$$

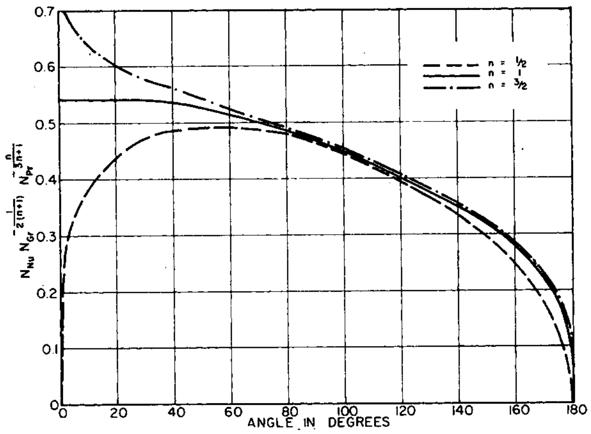

Fig. 4. The variation of the local rate of heat transfer along the surface of a horizontal cylinder.

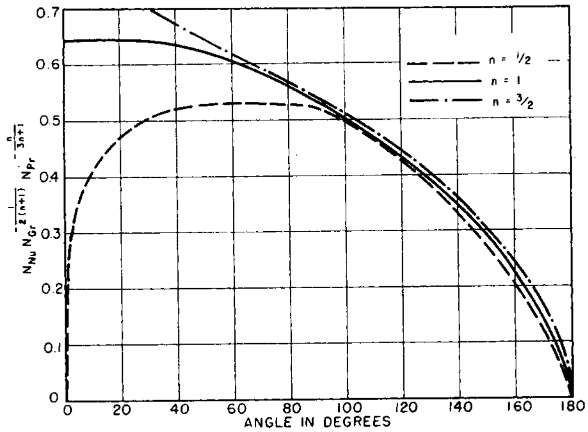

Fig. 5. The variation of the local rate of heat transfer along the surface of a sphere.

$$N_{Pr}^{\frac{n}{3n+1}} \frac{(r_1^n \sin \epsilon)^{\frac{1}{2n+1}}}{\left[\int_0^{x_1} r_1^{\frac{3n+1}{2n+1}} (\sin \epsilon)^{\frac{1}{2n+1}} dx_1\right]^{\frac{n}{3n+1}}}$$

where $\theta'(0)$ is a relatively insensitive function of n to be read off Figure 2 and where, for the purposes of this analysis, a fluid should be considered non-Newtonian only if the shear rate at the surface has a numerical value which is clearly within the region of applicability of the power-law given by Equation (1).

This expression for the local Nusselt number, which even for Newtonian fluids ($n = 1$) does not appear to have been derived before, is obtained from the exact asymptotic solution of the appropriate laminar boundary-layer equations and is applicable to any two-dimensional surface or a surface of revolution about an axis of symmetry. It is valid for all values of the non-Newtonian index parameter n and should be used when, as is usually the case, $N_{Gr}^{\frac{1}{n+1}} N_{Pr}^{\frac{2n}{3n+1}} > 10$, $N_{Pr} > 10$, and when the physical properties of the fluid may be assumed constant. It can also be modified, in those instances where the transfer occurs in a variable property fluid, by calculating $\theta'(0)$ from the solution of Equations (29) and (30); so that the formula for N_{Nu} shown above will still apply if it is multiplied by a correction factor which is independent of N_{Gr}, N_{Pr} and the surface geometry and is a function only of the particular fluid in question.

On the other hand the average Nusselt number $\overline{N_{Nu}}$ is given by

$$\overline{N_{Nu}} = C\, N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}}$$

where the constant C may readily be calculated by integrating the expression for the local N_{Nu} over the surface. Thus it would be expected that C would depend not only on the surface geometry but also on n. Surprisingly enough however numerical computations carried out for six surfaces and reported in Table 1 clearly show that C is only slightly affected by the value of n in the range $1/10 \leq n \leq 3/2$ which includes most of the power-law fluids of interest, and that furthermore exactly as with Newtonian fluids (7) it is rather insensitive to changes in the surface geometry. It is concluded therefore that, under the conditions stated earlier, average rates of heat transfer in laminar free convection may be computed with good accuracy from the expression

$$\overline{N_{Nu}} = C\, N_{Gr}^{\frac{1}{2(n+1)}} N_{Pr}^{\frac{n}{3n+1}}$$

where the constant C (approximately equal to 0.55) is only weakly dependent on the geometry of the surface and the numerical value of n.

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NOTATION

C = proportionality constant introduced in Equation (27a)
c_p = specific heat per unit mass
f = function introduced in Equation (22)
g = acceleration due to gravity
k = thermal conductivity
K, n = parameters of the power-model, Equation (1)
L = characteristic length
N_{Gr} = $\dfrac{\rho^2 L^{n+2}[\beta g(T_s - T_\infty)]^{2-n}}{K^2}$, the generalized Grashof number
N_{Nu} = Nusselt number
N_{Pr} = $\dfrac{\rho c_p}{k}\left(\dfrac{K}{\rho}\right)^{\frac{2}{1+n}} L^{\frac{1-n}{1+n}} [L\beta g(T_s - T_\infty)]^{\frac{3(n+1)}{2(1+n)}}$, the generalized Prandtl number
r_1 = r/L where r is the distance of a point on the surface from the axis of symmetry
T = temperature
T_s = temperature of the surface
T_∞ = temperature of the bulk of the fluid
U_c = characteristic velocity
u = velocity component along x
u_1 = dimensionless velocity component defined by Equation (13)
u_2 = dimensionless velocity component defined by Equation (18)
$\overline{u_2}$ = dimensionless velocity component defined by Equation (32)
v = velocity component along y
v_1 = dimensionless velocity component defined by Equation (13)
v_2 = dimensionless velocity component defined by Equation (18)
x = distance along the surface from the leading edge
x_1 = dimensionless distance x/L
X = dimensionless distance defined by Equation (32)
y = distance normal to the surface
y_1 = dimensionless distance y/L
y_2 = dimensionless distance defined by Equation (18)

Greek Letters

β = expansion coefficient of the fluid defined by Equation (5)
Γ = gamma function
ϵ = angle between the normal to the surface and the direction of the force of gravity (see Figure 1)
η = similarity variable defined by Equation (23)
θ = dimensionless temperature $\dfrac{T - T_\infty}{T_s - T_\infty}$
ρ = density
τ = the shear stress

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